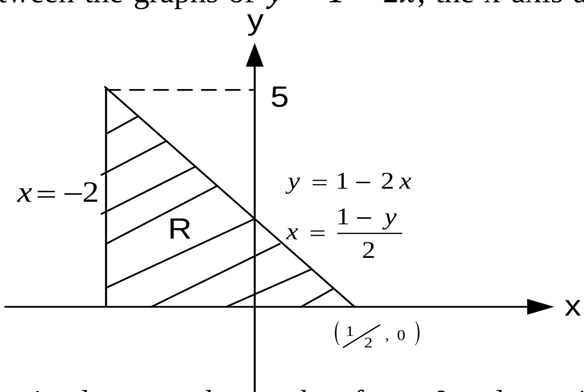
Adama Science And Technology University (ASTU)

School Of Applied Natural Science, Applied Mathematics Department.

Applied Mathematics II (Math 1102) *Solved Problems On Multiple Integrals*.

1. Let R be the region between the graphs of $y = 1 - 2x$, the x-axis and the line $x = -2$. Show that R is simple.

Solution

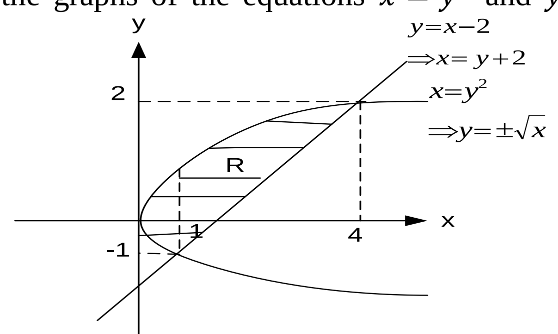


R is a vertically simple region between the graphs of $y = 0$ and $y = 1 - 2x$ for $-2 \le x \le \frac{1}{2}$.

i) R is a horizontally simple region between the graphs of $x = -2$ and $x = \frac{1-y}{2}$ for $0 \le y \le 5$.

ii) From i) and ii), we conclude that R is simple.

2. Let R be the plane region between the graphs of the equations $x = y^2$ and $y = x - 2$. Show that R is simple.



Solution

i) R is a horizontally simple region between the graphs of $x = y^2$ and $x = y + 2$ for $-1 \le y \le 2$.

ii) R is a vertically simple region between the graphs of $y = -\sqrt{x}$ and $y = +\sqrt{x}$ for $0 \le x \le 1$ and between $y = x - 2$ and $y = +\sqrt{x}$ for $1 \le x \le 4$.

iii) From i) and ii), we conclude that R is simple.

3. Evaluate each of the following iterated double integrals.

a) $\int_0^1 \int_x^{x+1} xy\, dy\, dx$

Solution

$$\int_0^1 \int_x^{x+1} xy\, dy\, dx = \int_0^1 \left[x \frac{y^2}{2} \Big|_x^{x+1} dx = \int_0^1 \frac{x}{2}\left[(x+1)^2 - x^2\right] dx \right]$$

$$= \int_0^1 \frac{x}{2}(2x+1)dx \quad = \int_0^1 \left(x^2 + \frac{1}{2}x\right)dx \quad = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)\Big|_0^1 \quad = \frac{7}{12}$$

b) $\int_1^3\int_0^3 \frac{2}{9+x^2}\,dx\,dy$

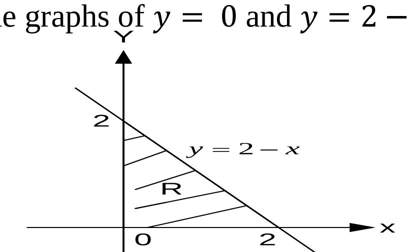
Solution $\int_1^3\int_0^3 \frac{2}{9+x^2}\,dx\,dy = \int_1^3 2\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^3\right]dy = \int_1^3 \frac{2}{3}\left[\tan^{-1}\left(\frac{3}{3}\right)-\tan^{-1}\left(\frac{0}{3}\right)\right]dy$

$$= \frac{2}{3}\int_1^3 \left[\tan^{-1}(1)-\tan^{-1}(0)\right]dy = \frac{2}{3}\int_1^3 \left(\pi/4-0\right)dy = \frac{\pi}{6}\int_1^3 dy = \frac{\pi}{3}$$

4. Evaluate $\iint_R x(x-1)e^{xy}\,dA$ if R the triangular region bounded by the lines $x=0$, $y=0$ and $x+y=2$.

Solution $f(x,y) = x(x-1)e^{xy}$

R is a vertically simple region between the graphs of $y = 0$ and $y = 2-x$ for $0 \le x \le 2$.

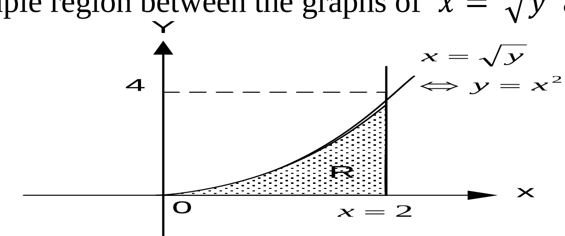


Thus, $\iint_R x(x-1)e^{xy}\,dA = \int_0^2\int_0^{2-x} x(x-1)e^{xy}\,dy\,dx = \int_0^2 \left[(x-1)e^{xy}\Big|_0^{2-x}\right]dx$

$$= \int_0^2 \left[(x-1)e^{2x-x^2}-(x-1)\right]dx = \int_0^2 (x-1)e^{2x-x^2}\,dx - \int_0^2 (x-1)\,dx = \left(-\frac{1}{2}e^{2x-x^2}\right)\Big|_0^2 - \left(\frac{x^2}{2}-x\right)\Big|_0^2 = 0$$

5. By reversing the order of integration, evaluate $\int_0^4\int_{\sqrt{y}}^2 \cos(x^3)\,dx\,dy$.

Solution: Note that it is impossible to evaluate the integral as it is. The plane region R is given as a horizontally simple region between the graphs of $x = \sqrt{y}$ and $x = 2$ for $0 \le y \le 4$.



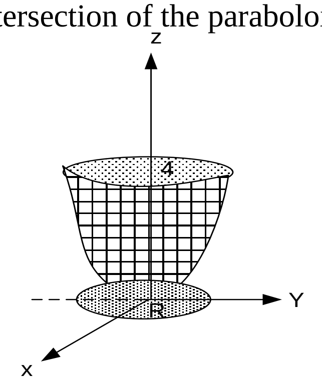
As a vertically simple region, R is a region between the graphs of $y = 0$ and $y = x^2$ for $0 \le x \le 2$.

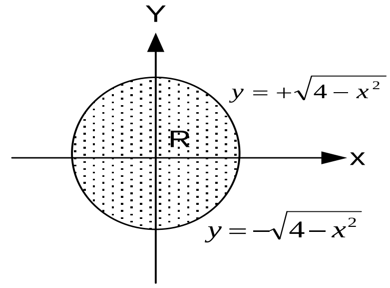
Thus, $\int_0^4\int_{\sqrt{y}}^2 \cos(x^3)\,dx\,dy = \int_0^2\int_0^{x^2} \cos(x^3)\,dy\,dx = \int_0^2\left[\cos(x^3)(y)\Big|_0^{x^2}\right]dx$

$$= \int_0^2 x^2\cos(x^3)\,dx = \frac{\sin(x^3)}{3}\Big|_0^2 = \frac{\sin 8}{3}$$

6. Find the volume of the solid region bounded by the paraboloid $z = x^2+y^2$ and the plane z = 4.

Solution The intersection of the paraboloid and the plane is the circle $x^2+y^2 = 4$ and this determines the region R.





$z = f(x,y) = x^2+y^2$ and R is a vertically simple region between the graphs of $y = -\sqrt{4-x^2}$ and $y = +\sqrt{4-x^2}$ for $-2 \le x \le 2$.

Volume $V = \iint\limits_R f(x,y)\,dA \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

$$= \int_{-2}^{2}\left[\left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\right] dx$$

$$= \int_{-2}^{2}\left[\left(x^2\left(\sqrt{4-x^2}\right) + \frac{1}{3}\left(4-x^2\right)\sqrt{4-x^2}\right) - \left(x^2\left(-\sqrt{4-x^2}\right) + \frac{1}{3}\left(4-x^2\right)\left(-\sqrt{4-x^2}\right)\right)\right] dx$$

$$= 2\int_{-2}^{2}\left(x^2\left(\sqrt{4-x^2}\right) + \frac{1}{3}\left(4-x^2\right)\sqrt{4-x^2}\right) dx \quad = 2\int_{-2}^{2}\left[\left(\frac{4}{3} + \frac{2}{3}x^2\right)\sqrt{4-x^2}\right] dx$$

Let $x = 2\sin\theta$ (trigonometric substitution) $\Rightarrow dx = 2\cos\theta\, d\theta$

Also $x = -2 \Rightarrow \theta = -\frac{\pi}{2}$ and $x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$$= 2\int_{-\pi/2}^{\pi/2}\left[\left(\frac{4}{3} + \frac{2}{3}(4\sin^2\theta)\right)2\cos\theta\right]2\cos\theta\, d\theta \quad = 8\int_{-\pi/2}^{\pi/2}\left(\frac{4}{3}\cos^2\theta + \frac{8}{3}\sin^2\theta\cos^2\theta\right)d\theta$$

$$= \frac{32}{3}\int_{-\pi/2}^{\pi/2}\left(\cos^2\theta + 2\sin^2\theta\cos^2\theta\right)d\theta \quad = \frac{32}{3}\int_{-\pi/2}^{\pi/2}\left(\cos^2\theta + 2(1-\cos^2\theta)(\cos^2\theta)\right)d\theta$$

$$= \frac{32}{3}\int_{-\pi/2}^{\pi/2}\left(3\cos^2\theta - 2\cos^4\theta\right)d\theta$$

$$= \frac{32}{3}\left[\frac{3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - 2\left(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta\right)\right]_{-\pi/2}^{\pi/2} \quad = 8\pi$$

7. Find the area of the plane region bounded by the graphs of $y = 3 - x^2$ and $y = 2|x|$.

Solution

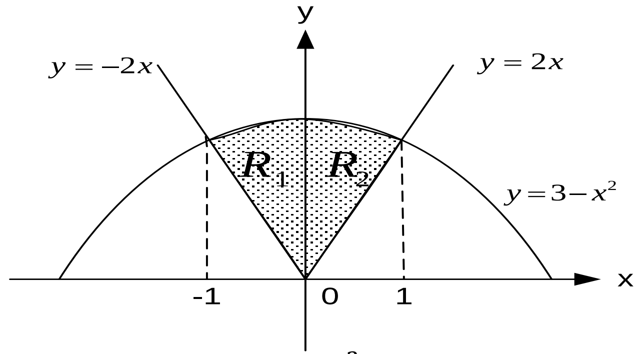
$R = R_1\, U\, R_2$

R_1 is a vertically simple region between the graphs of

$y = -2x$ and $y = 3 - x^2$ for $-1 \le x \le 0$.

R_2 is also a vertically simple region between the

graphs of $y = 2x$ and $y = 3 - x^2$ for $0 \le x \le 1$.



R_2 is also a vertically simple region between the graphs of $y = 2x$ and $y = 3 - x^2$ for $0 \le x \le 1$.

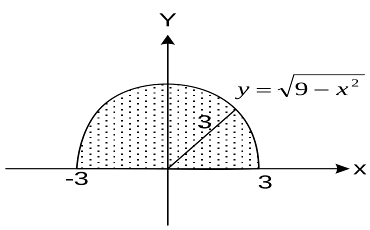
Area A of R is $A = \iint_R 1 dA = \iint_{R_1} 1 dA + \iint_{R_2} 1 dA = \int_{-1}^{0}\int_{-2x}^{3-x^2} dy\,dx + \int_{1}^{0}\int_{-2x}^{3-x^2} dy\,dx$

$$= \int_{-1}^{0}\left(y\Big|_{-2x}^{3-x^2}\right)dx + \int_{0}^{1}\left(y\Big|_{2x}^{3-x^2}\right)dx = \int_{-1}^{0}\left[\left(3-x^2\right)-(-2x)\right]dx + \int_{0}^{1}\left[\left(3-x^2\right)-(2x)\right]dx$$

$$= \int_{-1}^{0}\left(-x^2+2x+3\right)dx + \int_{0}^{1}(-x^2-2x+3)\,dx = \frac{5}{3}+\frac{5}{3} = \frac{10}{3}$$

8.Change the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$ to an iterated integral in polar coordinates and evaluate it

Solution: R is a vertically simple region between the graphs of $y = 0$ and $y = \sqrt{9-x^2}$ for $-3 \le x \le 3$.



$y = \sqrt{9-x^2}$, $y \ge 0 \Rightarrow y^2 = 9 - x^2$

$\Rightarrow y^2 + x^2 = 9 \Rightarrow r^2 = 9 \Rightarrow r = 3$

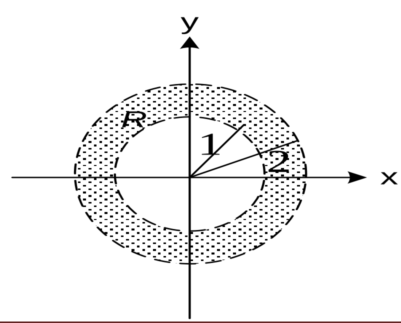
In polar coordinates, R is a region between the polar graphs of $r = 0$ *and* $r = 3$ for $0 \le \theta \le \pi$.

Thus, $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{0}^{\pi}\int_{0}^{3}\frac{1}{r}\cdot r\,dr\,d\theta = \int_{0}^{\pi}\int_{0}^{3} dr\,d\theta = 3\pi$

9. Let R be the region bounded by the circles $r = 1$ and $r = 2$ for $0 \le \theta \le 2\pi$.
   Evaluate $\iint_R \left(x^2 - y\right)dA$.

   Solution $\iint_R \left(x^2-y\right)dA = \int_{0}^{2\pi}\int_{1}^{2}\left[(r\cos\theta)^2 - (r\sin\theta)\right]r\,dr\,d\theta$

$$= \int_{0}^{2\pi}\int_{1}^{2}\left(r^2\cos^2\theta - r\sin\theta\right)r\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_1^2 \left(r^3\cos^2\theta - r^2\sin\theta\right) dr\,d\theta \quad = \int_0^{2\pi}\left[\left((\cos^2\theta)\frac{r^4}{4} - (\sin\theta)\frac{r^3}{3}\right)\Big|_1^2\right] d\theta$$

$$= \int_0^{2\pi}\left(\frac{15}{4}\cos^2\theta - \frac{7}{3}\sin\theta\right) d\theta \quad = \int_0^{2\pi}\left[\frac{15}{4}\left(\frac{1+\cos 2\theta}{2}\right) - \frac{7}{3}\sin\theta\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{15}{8}(1+\cos 2\theta) - \frac{7}{3}\sin\theta\right] d\theta = \frac{15}{8}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} + \frac{7}{3}\cos\theta\Big|_0^{2\pi} \quad = \frac{15}{4}\pi$$

10. Find the volume of the solid region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

Solution :Note that we have solved this problem by using rectangular coordinates and we have seen how tedious the calculation is.

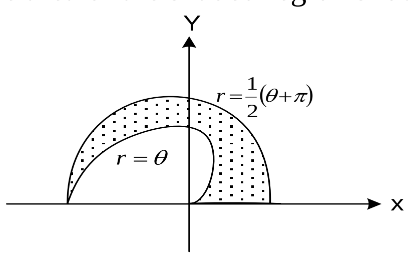
R is a region between the polar graphs of $r = 0$ and $r = 2$ for $0 \le \theta \le 2\pi$.

$$V = \iint_R f(x,y)\,dA = \iint_R \left(x^2+y^2\right) dA \quad = \int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta \quad = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \quad = \int_0^{2\pi}\left(\frac{1}{4}r^4\Big|_0^2\right) d\theta$$

$$= \int_0^{2\pi} 4\,d\theta \quad = 4\theta\big|_0^{2\pi} \quad = 8\pi$$

11. Find the area of the shaded region shown below.

Solution



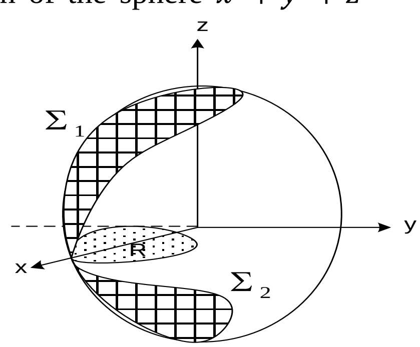
$r = 0\,, r = \frac{1}{2}(\theta + \pi)$

$0 \le \theta \le \pi$

Area A or R is $A = \iint_R 1\,dA = \int_0^{\pi}\int_0^{\frac{1}{2}(\theta+\pi)} r\,dr\,d\theta = \dfrac{\pi^3}{8}$

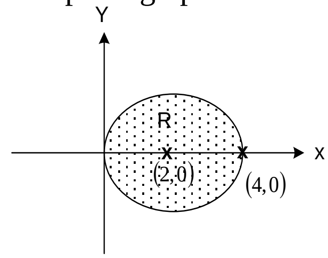
12. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ that lies inside the cylinder $x^2 - 4x + y^2 = 0$.

Solution



The surface Σ is the sum of the surface Σ_1 and Σ_2 which are of equal surface areas. Σ_1 is the graph of the function $f(x,y) = z = \sqrt{16 - x^2 - y^2}$ on R where R is the cylinder $x^2 - 4x + y^2 = 0$.

In polar coordinates, R is a region between the polar graphs of $r = 0$ and $r = 4\cos\theta$ for $-\pi/2 \le \theta \le \pi/2$.



Thus, the surface area S of Σ is given by $S = 2\iint\limits_R \sqrt{[fx(x,y)^2] + [fy(x,y)^2 + 1]}\, dA$

$f_x(x,y) = \frac{-x}{\sqrt{16-x^2-y^2}}$ and $f_y(x,y) = \frac{-y}{\sqrt{16-x^2-y^2}}$

$$S = 2\iint\limits_R \sqrt{\left(\frac{-x}{\sqrt{16-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{16-x^2-y^2}}\right)^2 + 1}\ dA$$

$$= 2\iint\limits_R \sqrt{\frac{x^2}{16-x^2-y^2} + \frac{y^2}{16-x^2-y^2} + 1}\ dA = 2\iint\limits_R \frac{4}{\sqrt{16-x^2-y^2}}\ dA$$

$$= 8\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta} \frac{1}{\sqrt{16-r^2}}\, r\,dr\,d\theta = 8\int_{-\pi/2}^{\pi/2}\left[\left(-\sqrt{16-r^2}\right)\Big|_0^{4\cos\theta}\right] d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2}\left[\left(\sqrt{16-r^2}\right)\Big|_{4\cos\theta}^{0}\right] d\theta = 8\int_{-\pi/2}^{\pi/2}\left[\left(\sqrt{16-0^2} - \sqrt{16-(4\cos\theta)^2}\right)\right] d\theta$$

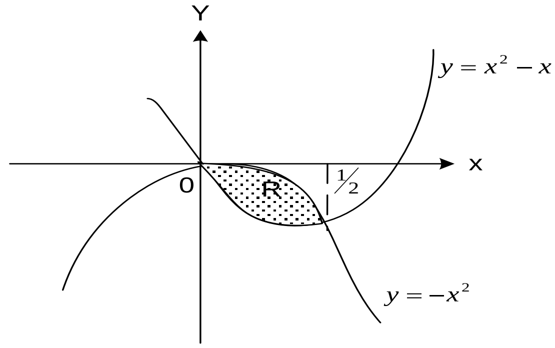
$$= 8\int_{-\pi/2}^{\pi/2}\left(4 - 4\sqrt{1-\cos^2\theta}\right) d\theta$$

$$= 32\int_{-\pi/2}^{\pi/2}\left(1 - |\sin\theta|1\right) d\theta \text{ , since } \sin^2\theta + \cos^2\theta = 1 \quad = 32\left[\int_{-\pi/2}^{0}(1 + \sin\theta)d\theta + \int_0^{\pi/2}(1 - \sin\theta)d\theta\right]$$

$$= 32\left[(\theta - \cos\theta)\Big|_{-\pi/2}^{0} + (\theta + \cos\theta)\Big|_0^{\pi/2}\right] = 32\ (\pi - 2)$$

13. Let R be the plane region between the graphs of $y = -x^2$ and $y = x^2 - x$. Find the moments of R about the x and the y axes. Also find the centroid of R.

Solution



R is a vertically simple region between the graphs of $y = x^2 - x$ and $y = -x^2$ for $0 \le x \le \frac{1}{2}$.

$$M_x = \iint_R y\, dA = \int_0^{1/2}\int_{x^2-x}^{-x^2} y\,dy\,dx = \int_0^{1/2}\left(\frac{y^2}{2}\Bigg|_{x^2-x}^{-x^2}\right)dx = \frac{1}{2}\int_0^{1/2}\left[\left(-x^2\right)^2 - \left(x^2-x\right)^2\right]dx$$

$$= \frac{1}{2}\int_0^{1/2}\left(2x^3 - x^2\right)dx = \frac{1}{2}\left(\frac{x^4}{2} - \frac{x^3}{3}\right)\Bigg|_0^{1/2} = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{3}\left(\frac{1}{8}\right)\right] = \frac{1}{16}\left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{1}{192}$$

$$M_y = \iint_R x\, dA = \int_0^{1/2}\int_{x^2-x}^{-x^2} x\,dy\,dx = \int_0^{1/2}\left((xy)\Bigg|_{x^2-x}^{-x^2}\right)dx = \int_0^{1/2} x\left[\left(-x^2\right) - \left(x^2-x\right)\right]dx$$

$$= \int_0^{1/2}\left(-x^3 - x^3 + x^2\right)dx = \int_0^{1/2}\left(-2x^3 + x^2\right)dx = \left(\frac{-x^4}{2} + \frac{x^3}{3}\right)\Bigg|_0^{1/2} = -\frac{1}{2}\left(\frac{1}{16}\right) + \frac{1}{3}\left(\frac{1}{8}\right)$$

$$= \frac{1}{8}\left(-\frac{1}{4} + \frac{1}{3}\right) = \frac{1}{96}$$

$$A = \iint_R dA = \int_0^{1/2}\int_{x^2-x}^{-x^2} dy\,dx = \int_0^{1/2}\left(y\Bigg|_{x^2-x}^{-x^2}\right)dx = \int_0^{1/2}\left[\left(-x^2\right) - \left(x^2-x\right)\right]dx = \int_0^{1/2}\left(-2x^2 + x\right)dx$$

$$= \left(-2\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_0^{1/2} = \frac{-2}{3}\left(\frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{24}$$

Hence , $\overline{x} = \frac{M_y}{A} = \frac{\frac{1}{96}}{\frac{1}{24}} = \frac{1}{4}$, $\overline{y} = \frac{M_x}{m} = \frac{-\frac{-1}{192}}{\frac{1}{24}} = -\frac{1}{8}$

Therefore, the center of gravity of R is at the point $(\overline{x}, \overline{y}) = \left(\frac{1}{4}, -\frac{1}{8}\right)$

14. Evaluate the iterated triple integral $\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\,dz\,dy\,dx$

Solution : $\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\,dz\,dy\,dx = \int_0^1\int_0^1\left((xy)\frac{z^2}{2}\Big|_{\sqrt{x^2+y^2}}^2\right)dy\,dx = \int_0^1\int_0^1 xy\left[2-\frac{(x^2+y^2)}{2}\right]dy\,dx$

$= \int_0^1\int_0^1\left(2xy - \frac{x^3y}{2} - \frac{xy^3}{2}\right)dy\,dx = \int_0^1\left[\left(xy^2 - \frac{x^3y^2}{4} - \frac{xy^4}{8}\right)\Big|_0^1\right]dx = \int_0^1\left(x - \frac{x^3}{4} - \frac{x}{8}\right)dx$

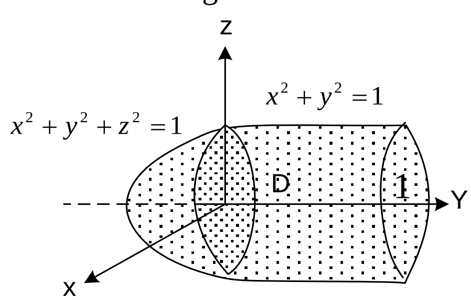
$= \int_0^1\left(\frac{7}{4}x - \frac{x^3}{4}\right)dx = \frac{3}{8}$

15. Evaluate the iterated triple integral $\int_0^2\int_0^y\int_0^{\sqrt{3}z}\frac{z}{x^2+z^2}\,dx\,dz\,dy$

Solution : $\int_0^2\int_0^y\int_0^{\sqrt{3}z}\frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_0^2\int_0^y\left[\tan^{-1}\left(\frac{x}{z}\right)\Big|_0^{\sqrt{3}z}\right]dz\,dy = \int_0^2\int_0^y\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(0)\right]dz\,dy$

$= \int_0^2\int_0^y\frac{\pi}{3}\,dz\,dy = \frac{\pi}{3}\int_0^2\left(z\Big|_0^y\right)dy = \frac{\pi}{3}\int_0^2 y\,dy = \frac{2\pi}{3}$

16. Evaluate $\iiint_D y\sqrt{1-x^2}\,dV$ where D is the region shown in the following figure.



Solution

It is sometimes advantageous to interchange the roles of x, y and z.In this example, we interchange the role of y and z. D is a solid region between the graphs of $y = -\sqrt{1-x^2-z^2}$ and $y = 1$ on R, where R is a simple region between the graphs of $z = -\sqrt{1-x^2}$ and $z = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

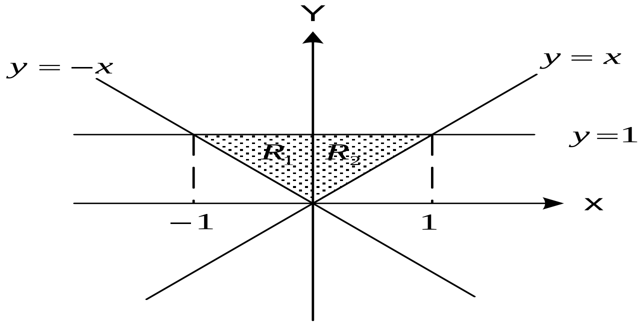
Therefore, $\iiint_D y\sqrt{1-x^2}\,dV = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-z^2}}^1 y\sqrt{1-x^2}\,dy\,dz\,dx = \frac{28}{45}$ (check !)

17. Evaluate $\iiint_D e^y\,dV$ where D is the solid region bounded by the planes $y = 1,\ z = 0,\ y = x,\ y = -x$

and $z = y$.

Solution : D is composed of two subregions D_1 and D_1.

D_1 is a solid region between the graphs of $z = 0$ and $z = y$ on R_1 where R_1 is a plane region between the graphs of $y = -x$ and $y = 1$ for $-1 \le x \le 0$, while D_2 is a solid region between the graphs of $z = 0$ and $z = y$ on R_2, where R_2 is a plane region between the graphs of $y = x$ and $y = 1$ for $0 \le x \le 1$.



Thus, $\iiint_D e^y\, dV = \iiint_{D_1} e^y\, dV + \iiint_{D_2} e^y\, dV \quad = \int_{-1}^{0}\int_{-x}^{1}\int_{0}^{y} e^y\, dz\, dy\, dx + \int_{0}^{1}\int_{x}^{1}\int_{0}^{y} e^y\, dz\, dy\, dx$

$= (e-2) + (e-2) \quad = 2e - 4$

18. Find the volume V of the solid region bounded above by the circular paraboloid $z = 4(x^2 + y^2)$ and below by the plane $z = -2$ and on the sides by the parabolic sheet $y = x^2$ and $y = x$.

Solution :D is a solid region between the graphs of $z = -2$ and $z = 4(x^2 + y^2)$ on R, where R is a vertically simple plane region between the graphs of $y = x^2$ and $y = x$ for $0 \le x \le 1$.

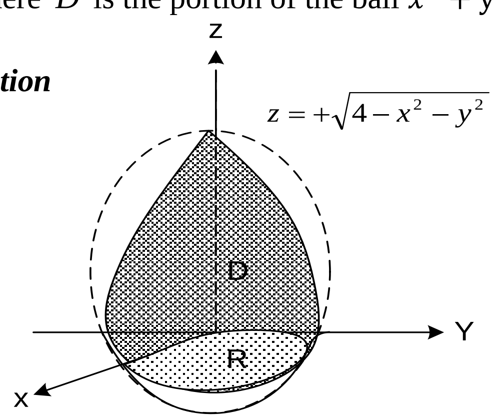
Thus, $V = \iiint_D 1\, dV = \int_0^1\int_{x^2}^{x}\int_{-2}^{4(x^2+y^2)} dz\, dy\, dx \;=\; \int_0^1\int_{x^2}^{x}\left[4(x^2+y^2) - (-2)\right] dy\, dx$

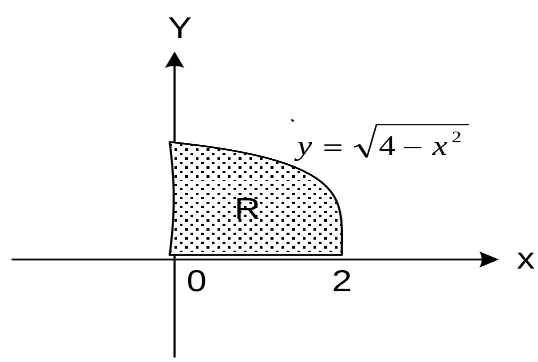
$= \int_0^1\int_{x^2}^{x}\left[4(x^2+y^2) + 2\right] dy\, dx = \dfrac{71}{105}$ (check!)

19. Express the triple integral as an iterated integral in cylindrical coordinates and evaluate it: $\iiint_D xz\, dV$,

where D is the portion of the ball $x^2 + y^2 + z^2 \le 4$ that is the first octant.

Solution





In polar coordinates, R is the region between the polar graphs of $r = 0$ and $r = 2$ for $0 \le \theta \le \frac{\pi}{2}$. Thus, in cylindrical coordinates, D is the region between the graphs of $z = 0$ and $z = \sqrt{4 - r^2}$ for (r, θ) in R.

Therefore, $\iiint\limits_D xz \; dV = \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} (r\cos\theta \, z) r \, dz \, dr \, d\theta = \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} (r^2 \cos\theta \; z) \, dz \, dr \; d\theta$

$$= \int_0^{\pi/2}\int_0^2 (r^2\cos\theta)\frac{z^2}{2}\Bigg|_0^{\sqrt{4-r^2}} dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^2 \cos\theta(4r^2 - r^4)dr\,d\theta = \frac{32}{15} \quad \text{(check!)}$$

20. Let D be the solid region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the planes $z = 0$, $x = \sqrt{3}\,y$ and $x = y$. Evaluate $\iiint\limits_D \sqrt{z}\, dV$.

Solution

We first determine the ranges of ρ , $\emptyset$ and θ. $0 \le \rho \le 4$

Since D is a first octant region , $0 \le \emptyset \le \frac{\pi}{2}$. $x = \sqrt{3}y \Rightarrow \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

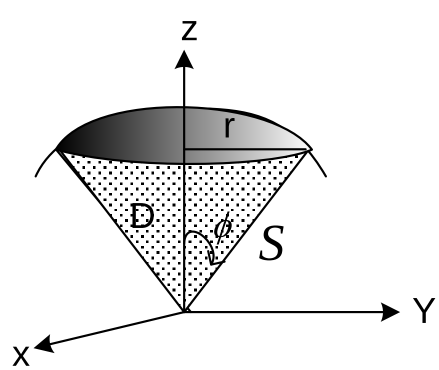
$x = y \Rightarrow \frac{y}{x} = 1 \Rightarrow \theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}(1) = \frac{\pi}{4}$

Hence , $\frac{\pi}{6} \le \theta \le \frac{\pi}{4}$ Now ,since $z = \rho \cos\emptyset$, we have

$$\iiint\limits_D \sqrt{z}\; dV = \int_{\pi/6}^{\pi/4}\int_0^{\pi/2}\int_0^4 \sqrt{\rho\cos\phi}\;\rho^2\sin\phi\; d\rho\; d\phi\, d\theta = \int_{\pi/6}^{\pi/4}\int_0^{\pi/2}\int_0^4 \rho^{5/2}\sin\phi\sqrt{\cos\phi}\;\; d\rho\; d\phi\, d\theta$$

$$= \int_{\pi/6}^{\pi/4}\int_0^{\pi/2}\left[\left(\frac{2}{7}\rho^{7/2}\Bigg|_0^4\right)\sin\phi\sqrt{\cos\phi}\right] d\phi\, d\theta = \frac{256}{7}\int_{\pi/6}^{\pi/4}\int_0^{\pi/2}\sqrt{\cos\phi}\,\sin\phi\;\; d\phi\, d\theta = \frac{128\pi}{63} \quad \text{(check!)}$$

21. Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the upper nape of the cone $z^2 = x^2 + y^2$.



Solution : $z^2 = \sqrt{x^2 + y^2} \Rightarrow z = r$

$\Rightarrow \rho\cos\emptyset = \rho\sin\emptyset$

$\Rightarrow \tan\,\emptyset = 1$

$\Rightarrow \emptyset = \frac{\pi}{4}$

$0 \le \rho \le 2$, $0 \le \emptyset \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$

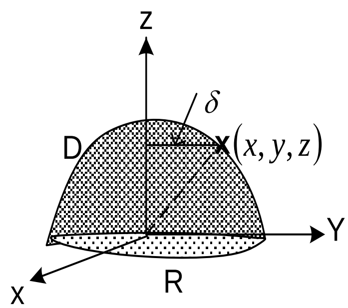
Volume $V = \iiint\limits_D dV$ Thus , $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

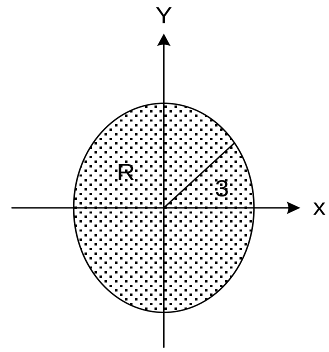
$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\left(\frac{\rho^3}{3}\right)\Bigg|_0^2 \sin\phi\right] d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\left(-\cos\phi \Bigg|_0^{\pi/4}\right) d\theta$$

$$= \frac{4\left(2-\sqrt{2}\right)}{3}\int_0^{2\pi} d\theta = \frac{8\left(2-\sqrt{2}\right)}{3}\pi$$

22. Find the center of gravity of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and below by the xy plane, $\delta(x,y,z)$ is equal to the distance form (x,y,z) to the z-axis.

Solution





D is the solid region, in spherical coordinates given by $0 \le \rho \le 3$, $0 \le \emptyset \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

$\delta(x,y,z) = \sqrt{x^2+y^2} = \sqrt{r^2} = r = \rho\sin\emptyset$.

Mass : $m = \iiint\limits_D \delta(x,y,z)dV = \iiint\limits_D \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (\rho\sin\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^3\sin^2\phi \, d\rho \, d\phi \, d\theta = \frac{81}{4}\int_0^{2\pi}\int_0^{\pi/2}\sin^2\phi \, d\phi \, d\theta = \frac{81}{4}\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{1-\cos 2\phi}{2}\right) d\phi \, d\theta$$

$$= \frac{81}{8}\int_0^{2\pi}\int_0^{\pi/2}(1-\cos 2\phi)\, d\phi \, d\theta = \frac{81}{8}\int_0^{2\pi}\left[\left(\phi - \frac{\sin 2\phi}{2}\right)\Bigg|_0^{\pi/2}\right] d\theta = \frac{81}{8}\int_0^{2\pi}\frac{\pi}{2}\, d\theta$$

$$= \frac{81}{8}\times\frac{\pi}{2}\times 2\pi = \frac{81}{8}\pi^2$$

We evaluate the three moments:

$$M_{xy} = \iiint\limits_D z\,\delta(x,y,z)dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (\rho\cos\phi)(\rho\sin\phi)\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^4\sin^2\phi\cos\phi \, d\rho \, d\phi \, d\theta = \frac{243}{5}\int_0^{2\pi}\int_0^{\pi/2}\sin^2\phi\cos\phi \, d\theta = \frac{243}{5}\int_0^{2\pi}\left[\left(\frac{\sin^3\phi}{3}\right)\Bigg|_0^{\pi/2}\right] d\theta$$

$$= \frac{81}{5}\int_0^{2\pi} d\theta \quad = \frac{162\,\pi}{5}$$

$$M_{xz} = \iiint_D y\,\delta(x,y,z)dV \quad = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} (\rho\sin\phi\,\sin\theta)\,(\rho\sin\phi)\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} \left(\rho^4\sin^3\phi\,\sin\theta\right) d\rho\; d\phi\; d\theta \;= \frac{243}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi\,\sin\theta\; d\phi\; d\theta$$

$$= \frac{243}{5}\int_0^{2\pi}\int_0^{\pi/2} \left(1-\cos^2\phi\right)\sin\phi\,\sin\theta\; d\phi\; d\theta \;= \frac{243}{5}\int_0^{2\pi}\left(\left(-\cos\,\phi + \frac{\cos^3\phi}{3}\right)\Bigg|_0^{\pi/2}\right)\sin\theta\; d\theta$$

$$= \frac{162}{5}\int_0^{2\pi}\sin\theta\, d\theta \;= \frac{162}{5}\,(-\cos\theta)\Big|_0^{2\pi} \;=\; 0$$

$$M_{yz} = \iiint_D x\;\delta(x,y,z)dV \quad = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} (\rho\sin\phi\;\rho\cos\phi)\,(\rho\sin\phi)\rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} \rho^4\sin^3\phi\,\cos\,\theta\, d\rho\; d\phi\; d\theta \;=\; 0 \quad \text{(Check !)}$$

Center of gravity $(\overline{x}\,, \overline{y},\ \overline{z}) = \left(\frac{M_{yz}}{m}\ , \frac{M_{xz}}{m}\ ,\ \frac{M_{xy}}{m}\right) = \left(\frac{0}{\frac{81}{8}\pi^2}\ , \frac{0}{\frac{81}{8}\pi^2}\ ,\ \frac{\frac{162}{5}\pi}{\frac{81}{8}\pi^2}\right) = \left(0\ ,\ 0\ ,\ \frac{16}{5\pi}\right)$